Journal of Statistical Physics, Vol. 33, No. 3, 1983

Solution of a Factorizable S-Matrix and an Asymmetric Eight-Vertex Model[1]

M. Doria[2,3] and R. Shankar[2,4]

Received June 2, 1983

We present here a solution to an asymmetric eight-vertex model using the recent correspondence between vertex models and factorizable S matrices.

KEY WORDS: Vertex models; factorizable S matrices.

1. INTRODUCTION

We present here a solution to an asymmetric eight-vertex model in two dimensions. While the model is a special case of the free-fermion models solved by Fan and Wu,(1) we still believe our solution of it interesting because it is not based on Fisher's dimer city approach(2) used in Ref. 1 but rather on the recently discovered connection between solvable vertex models on the one hand and certain "factorizable" S matrices on the other.(3,4) Thanks to this connection, the vertex problem reduces to the determination of a corresponding two-body S matrix, which is a lot simpler. For example, Baxter's eight-vertex model(5) corresponds to A. B. Zamolodchikov's Z_4 S matrix.(6) In this paper we derive a new S matrix and use it to solve the above-mentioned asymmetric eight-vertex model. We also show that there are no other physically interesting eight-vertex models. Since completing this work we have become aware of a similar (more exhaustive) *classification* of solvable six- and eight-vertex models.(7) This reference, however,

[1] Research supported in part by the U.S. Department of Energy under Contract No. DE-AC02-76ERO3075.

[2] J. W. Gibbs Laboratory, Yale University, New Haven, Connecticut 06511.

[3] Supported by Conselho Nacional de Desenvolvimento Cientifico e Tecnologico—Brasil.

[4] A. P. Sloan Fellow.

0022-4715/83/1200-0695$03.00/0

does not go into the explicit *solution* of the vertex model or the S matrix or their interconnection. Thus the basic contribution made here is as follows. (i) Provide one more instance where the S-matrix approach to a vertex model is seen to be correct and efficient, (ii) Provide yet another relativistic factorizable S matrix. While the underlying Lagrangian is not known in general, a special case reduces to the S matrix of the supersymmetric sine-Gordon theory found by Shankar and Witten.[8]

Since Refs. 3 and 4 review the vertex models, S matrices, and their interrelationship, we shall be content to give here a very schematic overview, starting with the vertex models.

Imagine an $N \times N$ lattice on the links or bonds of which reside some discrete degrees of freedom. For a given bond configuration on the lattice we associate a Boltzmann factor of the form

$$e^{-\beta E} = \prod_{i=1}^{N^2} \omega(i) \equiv \prod_{i=1}^{N^2} \omega_{\alpha_i \beta_i}^{\gamma_i \delta_i},$$

where $\alpha_i, \ldots, \delta_i$ label the states of the four bonds meeting at site i, i.e., label the "vertex" there, and $\omega(i)$ is the Boltzmann factor associated with that site. The goal is to calculate, for a given set of weights $\omega_{\alpha\beta}^{\gamma\delta}$,

$$\mathscr{Z} = \lim Z^{1/N^2} = \lim_{N\to\infty} \left[\sum_{\text{configs.}} \prod_{i=1}^{N^2} \omega(i) \right]^{1/N^2} \tag{1.1}$$

the partition function per site in the thermodynamic limit. (Note that in some cases, several ω's will be zero, i.e., some vertices will be forbidden. Several others may also be related by some symmetries.) We can write

$$Z = \operatorname{Tr} T^N \tag{1.2}$$

where the transfer matrix T is defined as follows. Consider a row of the lattice, shown in Fig. 1. Let α and α' denote collectively the states of the vertical bonds above and below the row; while i and i' describe the

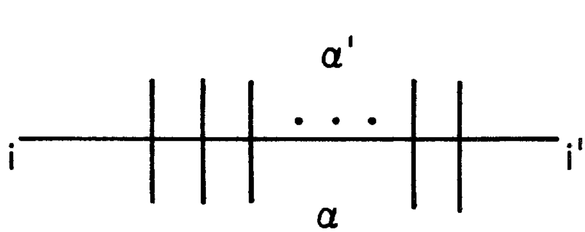

Fig. 1. A row of the lattice. The element $T_{\alpha\alpha'}$ (where α and α' are collective labels for the "initial" and "final" vertical arrows attached to the given row) is given by summing the product of Boltzmann factors over the horizontal states for the given choice α and α'.

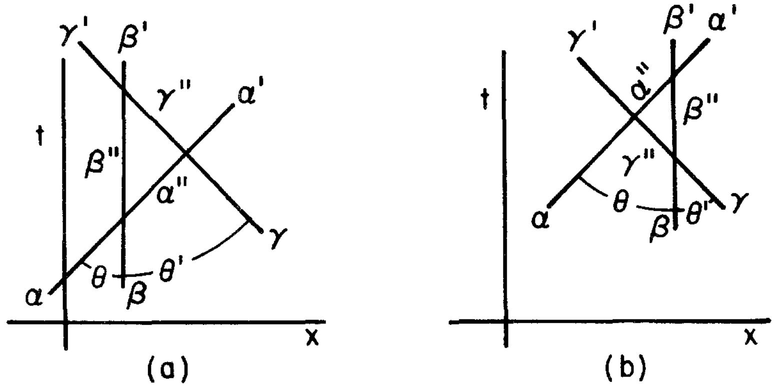

Fig. 2. (a) A three-body S matrix as a factorized product of three two-body S matrices. Here θ and θ' are the rapidity *differences*. (b) The amplitude, for a different choice of initial positions. Consistency requires that we get the same number in both cases.

horizontal bonds at the two ends for fixed value of α and α', sum over all allowed horizontal states with the appropriate weight and the periodic boundary condition $i = i'$. The resulting sum gives the element $T_{\alpha\alpha'}$, i.e., T acts on the vertical "spins," with horizontal spins summed over. One can then see that Eq. (1.2) defines a Z with toroidal boundary conditions. In the thermodynamic limit we simply need the dominant eigenvalue of T.

Now for the S matrices which describe two-body collisions in 1 + 1 dimensions. They are determined not from some given interaction Hamiltonian but as follows. We assume they obey the usual conditions of Lorentz invariance, crossing, unitary, etc., and in addition are elastic and factorizable—the N-body S matrix is a product of $N(N-1)/2$ two-body S matrices, one for each two-particle encounter. For example, Fig. 2a shows a three-body S matrix as a product of three two-body S matrices. Without changing of any of the initial momenta (or rapidities) we can get them to collide in a different sequence, as in Fig. 2b. Consistency of factorizability requires we get the same answer both ways.

Since each two-body collision is given by a matrix in internal (isospin) space, factorizability imposes some very stringent and overdeterminate set of constraints called Yang–Baxter or triangle equations. If $S_{\alpha\beta}^{\gamma\delta}(\theta)$ is the two-body matrix element for incoming particles (α, β) and out-going particles (γ, δ) with relative rapidity θ, then these equations are (see Fig. 2)

$$S_{\beta''\gamma''}^{\beta'\gamma'}(\theta')S_{\alpha''\gamma}^{\alpha'\gamma''}(\theta+\theta')S_{\alpha\beta}^{\alpha''\beta''}(\theta) = S_{\alpha''\beta''}^{\alpha'\beta'}(\theta)S_{\alpha\gamma''}^{\alpha''\gamma'}(\theta+\theta')S_{\beta\gamma}^{\beta''\gamma''}(\theta')$$

When solutions do exist, these equations determine the ratios of the two-body matrix elements at each rapidity. Unitarity then fixes the overall scale and a unique solution can be found if we impose some analyticity requirements. This is how S matrices have been found in the past.[5]

Here we present one more such S matrix describing collisions between two species of particles, call them bosons (b) and fermions (f). Both are neutral, but we require that the fermions be created or destroyed only in pairs. For a special value of one of the parameters, our solution becomes the S matrix of the supersymmetric sine-Gordon model, found earlier by Shankar and Witten.[(8)] In general, we do not know what the underlying field theory is, though our search will be briefly described.

We then proceed to exploit a certain connection between every factorizable S matrix and a related vertex model on a two-dimensional lattice to define and solve the latter, which in this case is an asymmetric eight-vertex model.

Now for the connection between the two problems. In his paper on the Z_4 S matrix,[(1)] A. B. Zamolodchikov showed that if one defines a vertex model with weights

$$\omega_{\alpha\beta}^{\gamma\delta} = \rho S_{\alpha\beta}^{\gamma\delta} \tag{1.3}$$

where $S_{\alpha\beta}^{\gamma\delta}(\theta, g)$ is a given two-body S matrix for rapidity θ and coupling constants g and ρ is any function independent of α, β, γ, or δ, then such a vertex model is *solvable*, in the sense that

$$\left[T(\theta, g), T(\theta', g) \right] = 0 \tag{1.4}$$

For example, his Z_4 S matrix defines Baxter's eight-vertex model.[(5)] Consequently, the S matrix we have found will also allow us to define a solvable vertex model, which happens to be an asymmetric eight-vertex model.

But we can go further and solve the model as follows. In Ref. 6, A.B.Z. also observed that the $\mathscr{G}$ for the Baxter model was very simply related to one of the Z_4 S matrix elements and said an explanation needed to be found. A brief explanation of this was provided by one of us in Ref. 10. A more detailed analysis and general result followed in Ref. 4, where it was shown that if we set up any solvable vertex model a la Zamolodchikov and let $\omega = S$ (instead of proportional to it), then

$$\mathscr{G}(s) = 1 \qquad \text{(in the P.R.)} \tag{1.5}$$

where the P.R. (principal region) corresponds to θ-imaginary, $0 < \operatorname{Im}\theta < \pi$, g such that all weights are positive. (The weights must also obey some other conditions. They do in this case, as described in Section 3.) Given this

[5] Reference 9 contains a comprehensive review. For a simpler one, see Exact S-matrices in two-dimensional field theories—A review, R. Shankar, Yale preprint No. C00 3075-199, 1979.

result, the answer for $\omega = \rho S$ follows by a simple rescaling:

$$\mathscr{Y}(\omega) = \rho \qquad \text{(in the P.R.)} \tag{1.6}$$

since scaling *all* weights by a factor ρ does the same to $\mathscr{Y}$. (This idea will be repeatedly used below.) Here is a (fictitious) illustrative example. Let there be just two amplitudes S_1, and S_2 in the S matrix and let their ratio, determined by the Yang–Baxter equations be

$$S_1(\theta) : S_2(\theta) = \cos(-i\theta) : \sin(-i\theta) \tag{1.7}$$

We now have the result

$$\mathscr{Y}(S_1, S_2) = 1 \tag{1.8}$$

Consider now the associated vertex problem with weights ω_1 and ω_2 in the ratio

$$\omega_1 : \omega_2 = 1 : x \tag{1.9}$$

We first choose $\theta = \theta_x$ such that

$$\omega_1 : \omega_2 = S_1 : S_2, \qquad \text{i.e.,} \quad \tan(-i\theta_x) = x \tag{1.10}$$

then

$$\begin{aligned} \mathscr{Y}(\omega_1, \omega_2) &= \omega_1 \mathscr{Y}(1, x) = \omega_1 \mathscr{Y}\left(1, \frac{S_2(\theta_x)}{S_1(\theta_x)}\right) \\ &= \frac{\omega_1}{S_1(\theta_x)} \mathscr{Y}(S_1(\theta_x), S_2(\theta_x)) \\ &= \frac{\omega_1}{S_1(\theta_x)} \qquad \text{using Eq. (1.8)} \end{aligned} \tag{1.11}$$

Thus the knowledge of $S(\theta_x)$ implies a knowledge of the solution to the associated vertex problem because of the result $\mathscr{Y}(s) = 1$.

(In general, however, we may have more independent ratios of weights than there are parameters (θ, g). In such a case a family of problems, but not all, can be solved and that too only in the P.R. Sometimes symmetries of $\mathscr{Y}$ may help us get the answer elsewhere.)

The reader should consult Ref. 4 for a careful analysis of the derivation leading to Eq. (1.5) and to see the connection with the related works of Stroganov,[11] Schultz,[12] A.B.Z.,[3] Baxter,[13] Pokrovsky and Bashilov.[14] In all these cases $\mathscr{Y}$ is computed by deriving some functional equations and looking for a solution with some analytic properties. In the derivation of the equation, however, boundary conditions are sometimes ignored (which is dangerous since unphysical regions with negative or complex weights necessarily enter the argument) or the analytic properties (known for S but not $\mathscr{D}$) are assumed. Even Ref. 4 is not completely satisfactory: A Lee–Yang-like theorem on the zeros of an eigenvalue of T is proven only for special cases and assumed in general.

In Section 2 we derive a new S matrix and study some of its features. In Section 3, we describe the corresponding eight-vertex model and its solution. *We shall see that along with Baxter's symmetric eight-vertex model, this exhausts the set of eight-vertex models with commuting transfer matrices.* (We do not include here models with unphysical weights.)

Upon solving our model using this trick, we searched the literature to see if any previously solved model coincided with ours, at least for some range of parameters, so that we could check our result. We found more than what we bargained for: our model is a special case of the free-fermion models solved by Fan and Wu[1] using Fisher's[2] dimer city trick.

2. THE S-MATRIX

Consider two species of particles, bosons (b) and fermions (f). We require that f's be created or destroyed only in pairs, though they are their own antiparticles, as are the bosons. The eight allowed amplitudes are shown in Fig. 3 along with their names. The incoming particles correspond to south and west bonds, while the outgoing ones, the other two. For example,

$$S_c(\theta) = \langle f(-\theta/2) b(\theta/2) \,|\, f(\theta/2) b(-\theta/2) \rangle \tag{2.1}$$

where $\pm\theta/2$ are the C.M. rapidities of the colliding particles. [We remind the reader that $p^\mu = (E, p) = m(\cosh\theta, \sinh\theta)$ defines the rapidity θ and that under a Lorentz transformation, $\theta \to \theta + \text{const.}$] We require (i) that $S(\theta)$ be meromorphic in θ, (ii) that $S(\theta)$ be real on the $\operatorname{Im}\theta$ axis, (iii) that under $\theta \to i\pi - \theta$, we get the crossed amplitude, i.e., the one with the vertical lines exchanged; thus the first four amplitudes are crossing symmetric while

$$\begin{gathered} S_c \xleftrightarrow[\theta \leftrightarrow i\pi - \theta]{} S_{\bar{d}} \\ S_{\bar{c}} \xleftrightarrow[\theta \leftrightarrow i\pi - \theta]{} S_d \end{gathered} \tag{2.2}$$

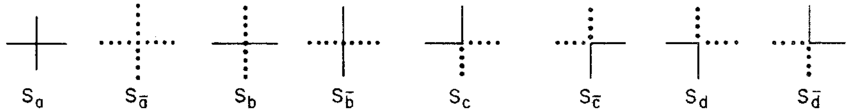

Fig. 3. The allowed amplitudes or vertices. The solid lines are f, the dotted ones b.

(iv) and that $S(\theta)$ obey unitarity:

$$S(\theta)S^T(-\theta) = I \tag{2.3}$$

where

$$\begin{array}{cccc} ff & bb & fb & bf \end{array}$$
$$S = \begin{bmatrix} S_a & S_{\bar{d}} & 0 & 0 \\ S_d & S_{\bar{a}} & 0 & 0 \\ 0 & 0 & S_b & S_{\bar{c}} \\ 0 & 0 & S_c & S_{\bar{b}} \end{bmatrix} \tag{2.4}$$

Finally, we require that the amplitudes obey the Yang–Baxter triangle equations to ensure factorizability. The details are relegated to the Appendix. *We find that these equations require*

(i) $$S_c = S_{\bar{c}} \tag{2.5a}$$

(ii) $$S_d = S_{\bar{d}} \tag{2.5b}$$

(iii) $$S_b = S_{\bar{b}} \tag{2.5c}$$

At this point we have a choice. If we let $S_a = S_{\bar{a}}$, we get the A.B.Z. solution for the S matrix or equivalently Baxter's ratio of weights, the symmetric eight-vertex case. If we insist $S_a \neq S_{\bar{a}}$, we get

$$S_a : S_{\bar{a}} : S_b : S_c : S_d$$
$$= 1 + \frac{sn}{cn\,dn} r : 1 - \frac{sn}{cn\,dn} r : -\frac{i}{f}\frac{sn}{cn\,dn} : \frac{1}{cn} : -ik\frac{sn}{dn} \tag{2.6}$$

where f is a real free parameter, while

$$r = -i\left(1 - k^2 + 1/f^2\right)^{1/2} \tag{2.7}$$

and sn, cn, dn are Jacobi's elliptic functions of modulus k[15] and argument

$$u = \frac{K'(k)\theta}{\pi} \tag{2.8}$$

Here $K'(k)$ is the complete elliptic integral of modulus $k' = (1 - k^2)^{1/2}$. For later use we add that $K(k') = K'(k)$. While f can be any real number here, for the use in the vertex model we require that

$$f > \frac{1}{\left[2k(k+1)\right]^{1/2}}, \qquad 0 < k \leqslant 1 \tag{2.9}$$

to ensure positivity of weights on the $\operatorname{Im}\theta$ axis for $0 < \operatorname{Im}\theta < \pi$ i.e., this is the P.R.

From the unitarity equation we get, upon eliminating S_c in favor of S_b:

$$S_b(\theta)S_b(-\theta) = \frac{sn^2(\theta K'/\pi)}{sn^2(\theta K'/\pi) + f^2 dn^2(\theta K'/\pi)} \tag{2.10}$$

while crossing requires

$$S_b(\theta) = S_b(i\pi - \theta) \tag{2.11}$$

The solutions to these equations are not unique: one can multiply any one solution by a function $\Phi(\theta)$ of the same class (meromorphic, etc.) obeying

$$\Phi(\theta)\Phi(-\theta) = 1 \tag{2.12a}$$

$$\Phi(\theta) = \Phi(i\pi - \theta) \tag{2.12b}$$

If, however, we seek a solution with no zeros or poles in the physical strip, $0 < \mathrm{Im}\,\theta < \pi$, $\Phi = 1$,[6] there is a unique "minimal" solution. It is obtained as follows:[16] Since $\ln S_b$ is analytic in the strip, consider a contour C surrounding it clockwise. We may write

$$\ln S_b(\theta) = \frac{1}{2\pi i} \oint_c \frac{dz}{\sinh(z-\theta)} \ln S_b(z)$$

The vertical pieces at infinity vanish owing to the sinh in the denominator. Using the fact that

$$\sinh(z + i\pi) = -\sinh(z)$$

$$S_b(z + i\pi) = S_b(-z) \qquad \text{(crossing)}$$

we get

$$\ln S_b(\theta) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dz}{\sinh(z-\theta)} \ln[S_b(z)S_b(-z)] \tag{2.13}$$

Using Eq. (2.10) we get finally

$$\ln S_b(\theta) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dz}{\sinh(z-\theta)} \ln\left[\frac{sn^2(zK'/\pi, k)}{sn^2(zK'/\pi, k) + f^2 dn^2(zK'/\pi, k)}\right] \tag{2.14}$$

The other amplitudes then follow from Eq. (2.6). *It is for this minimal solution that the result $\mathcal{Q}(S) = 1$ in the P.R. is true.*

Consider the special case $k = 1$. Using

$$sn(u, k=1) = \tanh u$$

$$cn(u, k=1) = dn(u, k=1) = \mathrm{sech}\, u$$

$$K'(\pi) = \pi/2$$

we get essentially the amplitudes for the supersymmetric sine-Gordon problem obtained by Shankar and Witten.[8] (We say essentially because some phases have to be chosen differently for the state vectors.) In this case the underlying Lagrangian is known:

$$\mathscr{L}(k=1)=\int\left[\frac{1}{2}(\partial_\mu\varphi)^2+\frac{i}{2}\bar{\psi}\partial\!\!\!/\psi-\frac{1}{2}\bar{\psi}\psi\cos\beta\varphi+\frac{1}{4\beta^2}\cos^2\beta\varphi\right]d^2x \tag{2.15}$$

where φ is a real scalar, ψ is a Majorana (self-conjugate) fermion, and β is a free parameter related to the f in Eq. (2.6). For $k\neq 1$ we do not know $\mathscr{L}(k\neq 1)$. (Since $cn[\varphi,(1-k^2)^{1/2}]=\cos\varphi$ when $k=1$, one may guess that $\cos\rightarrow cn$ in Eq. (2.11), but that does not work, and neither do many other guesses like $sn\,dn$, $(sn\,dn)^2$, cn^2, etc.)

3. SOLUTION TO AN ASYMMETRIC EIGHT-VERTEX MODEL

Consider now an asymmetric eight-vertex model with vertices in one-to-one correspondence with the eight amplitudes of the two-body S matrix. Let us parametrize the weights as follows:

$$\begin{bmatrix}\omega_a\\ \omega_{\bar{a}}\\ \omega_b\\ \omega_c\\ \omega_d\end{bmatrix}=\cos\frac{\theta K'}{2K}\cos\frac{(i\pi-\theta\ )K'}{2K}\begin{bmatrix}1+r\,sn/cn\,dn\\ 1-r\,sn/cn\,dn\\ (-i/f)sn/cn\,dn\\ 1/cn\\ -ik\,sn/dn\end{bmatrix} \tag{3.1}$$

where $r=-i(1-k^2+f^{-2})^{1/2}$, and the argument of the elliptic functions is $u=\theta K'/\pi$, while the modulus is k. The extra crossing symmetric factors of cosines are introduced to eliminate some singularities on the $\operatorname{Im}\theta=0$ or π axes along which we must integrate to get the free energy [Eq. (2.13)].

In Ref. 4 it is shown that the result $\mathscr{Q}(s)=1$ follows for $\theta=i\alpha$, $0<\alpha<\pi$ if the following is true:

(i) The S matrix is the minimal one.

(ii) All weights are positive (so Perron's theorem can be used). This we ensure by choosing

$$\begin{gathered}0<k<1\\ f^2>1/2k(k+1)\end{gathered} \tag{3.2}$$

(iii) At $\theta=0$,

$$S_{\alpha\beta}^{\gamma\delta}(0)\propto\delta_{\alpha\delta}\delta_{\beta\gamma}$$

This ensures that the logarithmic derivatives of T and the monodromy matrix are local operators, at $\theta = 0$, a crucial ingredient in the derivation. This is so here.

(iv) The largest eigenvalue of T, call it $\Lambda_B(\theta)$, is crossing symmetric. Since exchanging the vertical bonds in Fig. 2 gives us the transpose of T (or equivalently T^+ in the P.R.), and corresponds to $S_c \leftrightarrow S_d$, we have

$$T(i\pi - \theta) = T^+(\theta)$$

Using the fact that $\Lambda_B(\theta)$ is real by Perron's theorem, it follows that $\Lambda_B(\theta) = \Lambda_B^*(i\pi - \theta) = \Lambda_B(i\pi - \theta)$.

(v) There are no zeros or poles of $\Lambda_B(\theta)$ in the strip $0 \leqslant \mathrm{Im}\,\theta < \pi$. Since S_i are pole free in this region, so is Λ_B. However, it could have zeros. This is the assumption unproven here and in Ref. 4 for a general S-matrix-based vertex model. Given this assumption and the result $\mathscr{D}(s) = 1$, we get, by trivial rescaling, as in Eq. (1.1), the result

$$-\beta F = \ln\left[\frac{\omega_c}{S_c}\right] \tag{3.3}$$

Writing now a dispersion relation for ω_c/S_c, we get finally, upon setting $\theta = i\alpha$ [see Eqs. (2.13) and (2.14)]

$$\begin{aligned} -\beta F(\alpha, k, f) &= \frac{\sin\alpha}{\pi}\int_0^\infty \frac{dx\cosh x}{\cosh^2 x - \cos^2\alpha} \\ &\quad\times \ln\left\{\frac{\cos^2(\pi x/2K)\left[\cosh(\pi K'/K) + \cos(xK'/K)\right]\left[f^2 dn^2 u + sn^2 u\right]}{2f^2 cn^2 u\, dn^2 u}\right\} \end{aligned} \tag{3.4}$$

where $u = xK'/\pi$.

As mentioned earlier, this problem, whose eight weights obey

$$\omega_a\omega_{\bar{a}} + \omega_b\omega_{\bar{b}} = \omega_c\omega_{\bar{c}} + \omega_d\omega_{\bar{d}} \tag{3.5}$$

is a special case of the "free-fermion models" solved by Fan and Wu.[1] It has been verified numerically for many values of the parameters and analytically for others that the two results agree. It is, however, interesting that the present subset of the free fermion models which have commuting transfer matrices can be solved using the result $\mathscr{D}(s) = 1$.

If we set $\theta = i\pi/2$, $1/f = [k(k+1)]^{1/2}$, the weights correspond to that of an isotropic Ising model (if we transform from the spin to vertex form using a dual lattice). As k varies from 0 to 1, the temperature varies from 0 to the self-dual point.

It may be shown that the logarithmic derivative of T at $\theta = 0$ yields a local Hamiltonian H corresponding to an X–Y spin chain in a transverse field. The ground state energy of H is related to the derivative of F at $\theta = 0$. We do not present the results here since H may be easily solved using fermion operators.

4. CONCLUSIONS

First of all we have here a factorizable S matrix with two free parameters f and k. The underlying Lagrangian is unknown except at $k = 1$, which corresponds to the supersymmetric sine-Gordon theory.

The result $\mathscr{Q}(s) = 1$ then allows us to write down the free energy of an asymmetric eight-vertex model. Unfortunately, the result is not new since the weights obey the free-fermion condition and correspond to a previously solved case. It is, however, heartening to know that the result $\mathscr{Q}(s) = 1$ once again gives the correct solution. There are, however, no other eight-vertex models with commuting transfer matrices, as is shown in the Appendix.

APPENDIX

Here we briefly sketch the solution of the factorization equations. Consider first the reaction

$$(\alpha, \beta, \gamma) = (f, f, f) \to (\alpha', \beta', \gamma') = (f, f, f)$$

in Fig. 1. A sum over intermediate states is implied in Figs. 1a and 1b, there being two terms in each sum. In the present reaction, the terms with f's running around the loop cancel on the two sides and we get

$$S_d(\theta)S_{\bar{c}}(\theta + \theta')S_{\bar{d}}(\theta') = S_{\bar{d}}(\theta)S_c(\theta + \theta')S_d(\theta') \tag{A1}$$

In the future we will use a new notation in which S_i is replaced by i, so that (A1) reads

$$d(\theta)\bar{c}(\theta + \theta')\bar{d}(\theta') = \bar{d}(\theta)c(\theta + \theta')d(\theta') \tag{A2}$$

setting $\theta = \theta'$, we get

$$c(\theta) = \bar{c}(\theta)$$

Feeding this back to (A2), we get

$$\frac{d(\theta)}{\bar{d}(\theta)} = \frac{d(\theta')}{\bar{d}(\theta')} \tag{A3}$$

i.e.,

$$d(\theta) = \alpha\bar{d}(\theta) \tag{A4}$$

α being some constant. Consider next the reactions

$$fff \to fbb \quad \text{and} \quad bbf \to fff$$

The first gives

$$a(\theta)a(\theta+\theta')d(\theta') + d(\theta)c(\theta+\theta')\bar{a}(\theta') = c(\theta)d(\theta+\theta')a(\theta') + b(\theta)b(\theta+\theta')d(\theta') \tag{A5}$$

while the second, on exchanging θ and θ' gives

$$a(\theta)a(\theta+\theta')\bar{d}(\theta') + \bar{d}(\theta)c(\theta+\theta')\bar{a}(\theta') = \bar{b}(\theta)\bar{b}(\theta+\theta')\bar{d}(\theta') + c(\theta)\bar{d}(\theta+\theta')a(\theta') \tag{A6}$$

multiplying both sides by α, and using (A4) we get an equation with $\bar{d}$ replaced by d everywhere. Comparing that to (A5) we get

$$b(\theta)b(\theta+\theta')d(\theta') = \bar{b}(\theta)\bar{b}(\theta+\theta')d(\theta') \tag{A7}$$

Since $d(\theta') \neq 0$, we can cancel it to get

$$b(\theta)b(\theta+\theta') = \bar{b}(\theta)\bar{b}(\theta+\theta') \tag{A8}$$

Now we set $\theta' = 0$ and get

$$b(\theta) = \pm\bar{b}(\theta) \tag{A9}$$

For statistical mechanics $b = -\bar{b}$ is unphysical. (In the S-matrix case $b = -\bar{b}$ implies violation of parity. While this is possible, we do not explore the case $b \neq \bar{b}$ further.)

In what follows, we will also take $\bar{d} = d$, i.e., $\alpha = 1$. This is because none of the equations can determine α: either the equations contain only d's, with one d in each term (and likewise for $\bar{d}$), or if d and $\bar{d}$ occur, they occur together. Thus if we solve the case $d = \bar{d}$, we can generate another with $\tilde{d} = d/\sqrt{\alpha}$, $\tilde{\bar{d}} = \sqrt{\alpha}\, d$ obeying all the equations. In the eight-vertex problem this is no advance since only the combination $d\bar{d}$ enters the partition function. In the S-matrix case crossing [Eq. (2.2)] requires that if $S_c = S_{\bar{c}}$, then $S_d = S_{\bar{d}}$.

For the rest of the analysis, it is convenient to divide both sides of these equations by a factor $c(\theta)c(\theta+\theta')c(\theta')$. Ratios like $a(\theta)/c(\theta)$ will be denoted by capital letters, $A(\theta)$ in this case. We get the equations

$$DBB + \bar{A}D1 = D\bar{A}\bar{A} + A1D \tag{A10}$$

$$DB1 + \bar{A}DB = BD\bar{A} + 1BD \tag{A11}$$

$$1A1 + B1B = A1A + D\bar{A}D \tag{A12}$$

$$11B + BA1 = AB1 + DDB \tag{A13}$$

and four more with $A \leftrightarrow \bar{A}$ named (A$\overline{10}$) through (A$\overline{13}$), respectively. In these equations the arguments of the functions are θ, $\theta+\theta'$ and θ', respectively, going from left to right with 1 standing for the ratio c/c. Thus $1B\bar{A} \equiv 1B(\theta+\theta')\bar{A}(\theta')$, etc. Now set $\theta=0$ in (A13). We get

$$B(0)+\bar{A}(\theta')B(\theta') = B(\theta')\bar{A}(\theta') + B(0)D^2(\theta) \tag{A14}$$

Since we do not want $D^2 \equiv 1$ for all θ, we set

$$B(0)=0 \tag{A15}$$

Now set $\theta'=0$ in Eq. (A11), and recall that $B(0)=0$. Then

$$D(\theta)\left[1-\bar{A}(0)\right] = D(0)$$

If we do not want $D(\theta)=\text{const}$, we must choose

$$D(0)=0 \tag{A16}$$

$$\bar{A}(0)=1 \tag{A17}$$

From Eq. (A11), we get also

$$A(0)=1 \tag{A18}$$

The next step is to follow Zamolodchikov,[6] differentiate Eqs. (A10)–(A13) with respect to θ and set $\theta=0$. Letting $\alpha_A = \dot{A}(0) = dA/d\theta|_{\theta=0}$, etc., we get

$$\dot{D} = \alpha_D[\bar{A}^2 - B^2] + (\alpha_A - \alpha_{\bar{A}})D \tag{A19}$$

$$\alpha_D B + \alpha_{\bar{A}} DB + \dot{D}B = \alpha_B D\bar{A} + \dot{B}D \tag{A20}$$

$$\dot{A} = \alpha_A A - \alpha_B B + \alpha_D \bar{A} D \tag{A21}$$

$$\dot{B} = \alpha_B A - \alpha_A B - \alpha_D BD \tag{A22}$$

and four more with $\alpha_A \leftrightarrow \alpha_{\bar{A}}$, $A \leftrightarrow \bar{A}$, labeled (A$\overline{19}$) to (A$\overline{22}$). Consider (A22) minus (A$\overline{22}$). We get

$$A-\bar{A} = \frac{\alpha_A - \alpha_{\bar{A}}}{\alpha_B} B \tag{A23}$$

Now consider (A19) minus (A$\overline{19}$):

$$A^2 - \bar{A}^2 = 2\frac{\alpha_A - \alpha_{\bar{A}}}{\alpha_D} D \tag{A24}$$

It now becomes convenient to work with the combinations

$$h_1 = \frac{A-\bar{A}}{2}, \qquad h_2 = \frac{A-\bar{A}}{2}$$

(with α_1 and α_2 defined as usual). Some simple manipulations tell us that

$\alpha_2 = 0$ and give us an equation for h_2 alone:

$$\dot{h}_2^2 = \left(1 - h_2^2\right)\left(\alpha_B^2 - \alpha_1^2 + \alpha_D^2 h_2^2\right)$$

which is of the elliptic type.

We skip the rest of the details and mention just the following:

(i) There are four α's in the equations. One of them, α_2, can be shown to be zero. One more is not really a variable since changing it only changes θ by a factor. In the S-matrix case, it is chosen to ensure crossing. Thus we have a two-parameter family of solutions at each θ.

(ii) We have divided all the weights by a factor $cn(\theta K'/\pi, k)$ to give them the desired crossing properties.

REFERENCES

1. C. Fan and F. Y. Wu, *Phys. Rev.* **B2**:723 (1970).
2. M. E. Fisher, *J. Math. Phys.* **7**:1776 (1966).
3. A. B. Zamolodchikov, *Sov. Sci. Rev.* **2** (1980).
4. R. Shankar, *J. Stat. Phys.* **29**:649 (1982).
5. R. J. Baxter, *Ann. Phys.* (*N.Y.*) **70**:193 (1970); R. J. Baxter, *J. Stat. Phys.* **8**:25 (1973).
6. A. B. Zamolodchikov, *Commun. Math. Phys.* **69**:165 (1979).
7. K. Sogo, M. Uchinami, Y. Akutsu, and M. Wadati, *Prog. Theor. Phys.* **68**:508 (1982).
8. R. Shankar and E. Witten, *Phys. Rev. D* **17**:213 (1978).
9. A. B. Zamolodchikov and Al. B. Zamolodchikov, *Ann. Phys.* (*N.Y.*) **120**:253 (1979).
10. R. Shankar, *Phys. Rev. Lett.* **47**:1177 (1981).
11. Y. A. Stroganov, *Phys. Lett.* **74A**:116 (1979).
12. C. L. Schultz, *Phys. Rev. Lett.* **46**:629 (1981).
13. R. J. Baxter, *J. Stat. Phys.* **28**:1 (1982).
14. S. V. Pokrovsky and Yu. A. Bashilov, *Commun. Math. Phys.* **84**:103 (1982).
15. I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, 5th ed. (Academic Press, New York, 1980).
16. M. Karowski, H. J. Thun, T. T. Troung, and P. Weisz, *Phys. Lett.* **62B**:221 (1977).